Asymptotic Maturity Behavior of Single Factor Heath-Jarrow-Morton Term Structure Models: A Note

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ABSTRACT

Within the single factor Heath-Jarrow-Morton framework an analysis of three term structure properties exhibited by several existing models is conducted. Necessary conditions for these properties are determined and a discussion regarding the plausibility of certain term structure evolutions suggests that these conditions should not be substantially weakened. An analysis of the asymptotic behavior of the term structure's evolution as a consequence of these conditions indicates that: i) the dynamics of the infinitely maturing forward rate are locally deterministic, and ii) the infinitely maturing forward rate cannot fall over time (which is consistent with the result of Dybvig-Ingersoll-Ross (1996)).

I. INTRODUCTION

Most single factor continuous time models of the term structure dynamics can be, or are, parameterized in the single factor version of the Heath-Jarrow-Morton (1992) (hereafter HJM) framework where one source of noise introduces all uncertainty. Examples of models that fall into this category include those from Vasicek (1977), the Cox-Ingersoll-Ross (1985b) model, Hull-White (1990,1993), the Duffie (1992) "affine class of models", the Ritchken-Sankarasubramanian (1995) class of models, and those models arising from Jeffrey (1995) and Jeffrey (1997).

Within the single factor HJM framework the initial forward rate curve is exogenously specified and the intertemporal transitions of the whole forward rate curve are dictated by the specified forward rate volatility structure (the volatility of each forward rate with different maturity). The initial forward rate curve and forward rate volatility structure will hereafter be referred to as "the HJM parameters". Since the behavior of future term structures (those after the specified initial term structure) is controlled by the choice of HJM parameters, these parameters deserve consideration. For example, the HJM constant forward rate volatility structure model (given as an example in HJM (1992)) has the property that the drift component in the forward rate dynamics increases monotonically to infinity as maturity increases[1]. This indicates that, even over a very short period of time, long maturity forward rates are always expected to increase over time and further, the infinite maturity forward rate will increase by an infinite amount. Consequently such a model has the potential of providing plausible interest rate levels in the short term but unrealistic high long term interest rates. In contrast, the HJM exponential decay forward rate volatility structure model (also provided as an example in HJM (1992)) does not possess this property. More specifically, the drift component in the forward rate dynamics decreases to zero as maturity increases indicating that the long term forward rates will be approximately the same over time.

[1] This will become clear in Section III.

Motivated by the disparate long term maturity characteristics resulting from these two volatility structures, this paper considers some term structure properties exhibited by several existing models. Necessary conditions are determined for the HJM parameters as a consequence of imposing these properties and the dynamic behavior of long term forward rates is discussed. After considering "plausible" economic behavior, in conjunction with the asymptotic behavior of the term structure, it is argued that these necessary conditions should not be substantially weakened. The striking implications of this analysis are i) the volatility structure should be zero asymptotically in maturity, and ii) the infinite maturity forward rate never decreases over time. The latter result is consistent with a similar result obtained in Dybvig-Ingersoll-Ross (1996).

The remainder of this paper is structured as follows. To set the framework, Section II summarizes the single factor version of the HJM (1992) framework. The consequences of three term structure properties exhibited by several existing models are discussed in Section III, and Section IV concludes.

II. SUMMARY OF THE SINGLE FACTOR HJM FRAMEWORK

To set notation, let $P(t,T)$ denote the price of a one dollar face value, default free, zero coupon bond at time t which will mature at time T. The instantaneous forward rate at time t for date T, denoted $f(t,T)$, is defined through the relation

$$\ln\left[P(t,T)\right] = -\int_t^T f(t,v)\, dv\,.$$

Further the instantaneous forward rate at time t for date t is the instantaneous spot interest rate at time t, denoted $r(t)$; that is $r(t) = f(t,t)$.

The HJM (1992) framework considers the term structure in terms of forward rates and the uncertain evolution of each forward rate is modeled by fixing a maturity date T and then characterizing the evolution of $f(t,T)$ with the stochastic differential equation

$$df(t,T) = \alpha(\omega,t,T)\,dt + \gamma(\omega,t,T)\,dz(t) \qquad \textbf{(1)}$$

where $z(t)$ is a Wiener process, ω indicates the possible dependence on the term structure's realization up to time t, and

$$\alpha(\omega,t,T) = \gamma(\omega,t,T)\left(\int_t^T \gamma(\omega,t,v)\,dv + \lambda(\omega,t) \right),$$

which results from the no-arbitrage condition in the bond market where $\lambda(\omega,t)$ represents the market price of risk for the single source of uncertainty in the economy[2]. The function $\gamma(\omega,t,T)$ represents the volatility of the T-maturity forward rate at time t and is referred to as the "forward rate volatility structure". It is easily shown that the volatility of the spot interest rate is $\gamma(\omega,t,t)$ [3] and, following the usual convention, the sign of the spot interest rate volatility is chosen to be positive[4]. Throughout this paper the integral form of equation (1) is frequently required and hence is provided below;

$$f(t,T) = f(0,T) + \int_0^t \alpha(\omega,s,T)\,ds + \int_0^t \gamma(\omega,s,T)\,dz(s)\,. \qquad \textbf{(2)}$$

HJM also provide sufficient regularity conditions to ensure that the above framework is valid[5] and

[2] The market price of risk for the single source of uncertainty is the required excess return on bond prices above the risk free interest rate per unit of risk which is measured by the volatility of bond price returns; see Vasicek (1977), Cox-Ingersoll-Ross (1985b), or HJM (1992).

[3] See Jeffrey (1995), Section II.

[4] Technically the sign of the spot interest rate volatility can be positive or negative without any effect on the resulting term structure model. This is because a Wiener process is symmetric about zero.

[5] These regularity conditions are weak in the sense that they ensure enough structure for the term structure's evolution and the market price of risk to be well defined.

throughout this paper these regularity conditions are implicitly assumed. An element of the HJM framework that is of importance in this paper is that time t and the maturity date T are finite. However, given that t and T can be arbitrarily large, the asymptotic maturity behavior of the term structure's evolution is studied in this paper by taking the results of the HJM framework as given and then considering limiting behavior.

III. IMPLICATIONS OF PROPERTIES TYPICALLY OBSERVED IN EXISTING TERM STRUCTURE MODELS

In this section three properties exhibited by several existing term structure models are discussed in relation to the HJM parameters. This discussion provides insights regarding the implications of certain HJM parameter choices, in particular the implications for the dynamic behavior of long term forward rates. The three properties considered are the following;

***Property 1**: At every point in time t, and every maturity date T, the forward rate curve* $f(t,T)$ *is at least once differentiable in maturity T.*

***Property 2**: There is always uncertainty in the dynamics of all forward rates.*

***Property 3**: At every point in time the infinite maturity forward rate,* $\lim_{T\to\infty} f(t,T)$, *exists and is finite.*

From a modeling perspective the above properties are appealing since i) Property 1 simplifies analysis by permitting differentiation with respect to maturity and conforms to the usual practice of approximating the term structure via some smooth interpolation[6], ii) Property 2 implies that the

[6] For example, the spline interpolations by McCulloch (1971, 1975) and Chen (1987), and the functional approximation by Nelson and Siegel (1987).

future is always uncertain, and iii) Property 3 allows for analytic tractability of long maturity forward rates. Given the closed form representations of the Vasicek (1977) and CIR (1985b) models, it is easily verified that they exhibit the above properties. In addition, the HJM exponential decay volatility structure model also exhibits the above properties providing the chosen initial term structure satisfies properties 1 and 3. However, the HJM constant forward rate volatility structure model violates Property 3 even if the initial forward rate curve satisfies both properties 1 and 3[7]. Necessary conditions are now determined for the HJM parameters as a result of imposing these properties.

Property 1 states that the forward rate curve at every point in time is differentiable with respect to maturity, that is $\partial f(t,T)/\partial T$ exists for all T. Further, the dynamic representation of $\partial f(t,T)/\partial T$ is[8]

$$\frac{\partial f(t,T)}{\partial T} = \frac{\partial f(0,T)}{\partial T} + \int_0^t \frac{\partial \alpha(\omega,s,T)}{\partial T}\, ds + \int_0^t \frac{\partial \gamma(\omega,s,T)}{\partial T}\, dz(s) \; .$$

Consequently both the initial forward rate curve and the forward rate volatility structure generally must be differentiable with respect to maturity. This is formalized by stating that the following two necessary conditions must hold if Property 1 is satisfied[9],

Condition C1: $\partial f(0,T) \,/\, \partial T$ *exists and is finite for all* $T \in [\,0,\infty)$*;*

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[7] This becomes clear shortly since this volatility structure violates a necessary condition for Property 3 (in particular Condition C5).

[8] This results directly from equation (2).

[9] Note that the existence of $\partial\,\alpha(\omega,t,T)\,/\,\partial\,T$ is implied by the existence of $\partial\,\gamma(\omega,t,T)\,/\,\partial\,T$ and $\lambda(\omega,t)$.

Condition C2: *At every point in time* $t \in [0,\infty)$, $\partial \gamma(\omega,t,T) \,/\, \partial T$ *exists and is finite for all* $T \in [0,\infty)$.

To determine the implications of the second property, note that uncertainty is introduced into the HJM framework via the Wiener process $z(t)$ in equation (1). If for a particular maturity date T the coefficient of the incremental Wiener process $dz(t)$ is non-zero at time t then uncertainty exists in the evolution from $f(t,T)$ to $f(t+dt,T)$, otherwise the dynamics of $f(t,T)$ are locally deterministic (locally riskless) at time t. Hence the following condition can be taken as the definition of Property 2,[10]

Condition C3: *At every point in time* $t \in [0,\infty)$, $\gamma(\omega,t,T) \neq 0$ *for all* $T \in [t,\infty)$.

In order to analyze the long term characteristics of the HJM framework, which is constructed within a finite time horizon and finite maturity date setting, the results of the framework are taken as given and then limits are considered over time and maturity. In order to determine implications

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[10] It is interesting to note that if the term structure is quoted via the yield curve $y(t,T)$, where the yield curve can be calculated from the forward rate curve via $y(t,T) = \int_t^T f(t,v)\, dv$, then ensuring that the yield curve volatility structure is non-zero everywhere does <u>not</u> imply that the forward rate volatility structure is non-zero everywhere. This is demonstrated by the following. Consider modeling the term structure's evolution via

$$dy(t,T) = \alpha_y(\omega,t,T)\, dt + \gamma_y(\omega,t,T)\, dz(t) .$$

Given the relationship between yields and forward rates, the corresponding relationship between yield and forward rate volatility structures is

$$\gamma_y(\omega,t,T) = \frac{1}{T-t}\int_t^T \gamma(\omega,t,v)\, dv, \quad \text{or equivalently} \quad \gamma(\omega,t,T) = \gamma_y(\omega,t,T) + \frac{\partial \gamma_y(\omega,t,T)}{\partial T}(T-t).$$

Now consider the yield curve volatility structure $\gamma_y(\omega,t,T) = \sigma e^{-\kappa(T-t)}$, where $\kappa > 0$. This is never zero implying that uncertainty is always present in yield curve dynamics. This volatility structure is equivalent to the forward rate volatility structure $\gamma(\omega,t,T) = \sigma(1-\kappa(T-t))\, e^{-\kappa(T-t)}$ where $\kappa > 0$. Expressing the volatility structure in this form shows that the volatility of the forward rate $f(t,t+1/\kappa)$ is always zero implying $f(t,t+1/\kappa)$ is always locally deterministic.

of Property 3 consider the dynamic representation of the infinite maturity forward rate[11],

$$\lim_{T\to\infty} f(t,T) \;=\; \lim_{T\to\infty}\left[f(0,T) \;+\; \int_0^t \alpha(\omega,s,T)\,ds \;+\; \int_0^t \gamma(\omega,s,T)\,dz(s) \right]$$

$$\text{where}\quad \alpha(\omega,s,T) \;=\; \gamma(\omega,s,T)\left[\int_s^T \gamma(\omega,s,v)\,dv \;+\; \lambda(\omega,s) \right].$$

Consequently for the dynamics of $\lim_{T\to\infty} f(t,T)$ to be well defined it is necessary that $\lim_{T\to\infty} f(0,T)$, $\lim_{T\to\infty} \alpha(\omega,t,T)$, and $\lim_{T\to\infty} \gamma(\omega,t,T)$ all exist and are finite at every point in time. Hence the following necessary conditions for Property 3[12],

Condition C4: $\lim_{T\to\infty} f(0,T) \;=\; f(0,\infty)$ *which is a finite constant;*

Condition C5: $\lim_{T\to\infty} \gamma(\omega,t,T)\int_t^T \gamma(\omega,t,v)\,dv \;=\; B(\omega,t)$

for every point in time $\mathrm{t} \in [0,\infty)$ *where* $\mathrm{B}(\omega,t)$ *is finite valued for all* $\mathrm{t} \in [0,\infty)$ *and may depend on the term structure's realization up to time t.*

The asymptotic maturity implications of the last property are striking. The most notable implication is that Condition C5 implies $\lim_{T\to\infty} \gamma(\omega,t,T) \;=\; 0$. Consequently Property 2, which is intuitively appealing in the sense that there should always be uncertainty in term structure movements, cannot hold for the infinite maturity forward rate if Property 3 is to be satisfied. Given conditions C4 and C5 are satisfied, the dynamics of the infinite maturity forward rate (denoted

[11] This representation obtains directly from equation (2) assuming the infinite maturity forward rate exists.

[12] Given the market price of risk is finite at every point in time, and given $\lim_{T\to\infty} \gamma(\omega,t,T)$ is finite, then Condition C5 is necessary for $\lim_{T\to\infty} \alpha(\omega,t,T)$ to be finite. Further, Condition C5 is sufficient for $\lim_{T\to\infty} \gamma(\omega,t,T)$ to be finite.

$f(t,\infty)$) are locally deterministic and can be expressed as[13]

$$f(t,\infty) \;=\; f(0,\infty) \;+\; \lim_{\tau \to \infty} \int_0^t \left[\gamma(\omega,s,s+\tau) \int_0^\tau \gamma(\omega,s,s+v)\, dv \right] ds\,. \qquad \textbf{(3)}$$

There are some obvious classes of models for which the behavior over time of the infinite maturity forward rate can be determined. First, the class of deterministic forward rate volatility structures implies that $f(t,\infty)$ can be calculated directly from equation (3). Second, models which have a volatility structure with the property $|\int_0^\infty \gamma(\omega,s,s+v)\, dv| < \infty$ for all $s \in [0,t]$ imply $f(t,\infty) = f(0,\infty)$[14]. Finally, consider a framework where the forward rate curve at time t is a function of the spot interest rate at time t, time and maturity only. For convenience the time t forward rate curve is denoted $f(r(t),t,T)$ and it is also assumed that $\partial f(r(t),t,T) \,/\, \partial t$ and $\partial^2 f(r(t),t,T) \,/\, \partial r(t)^2$ always exist. Examples of such frameworks include Vasicek (1977), Hull and White (1990) and Jeffrey (1995). In such a setting the infinite maturity forward rate is totally deterministic[15] rather than just locally deterministic.

An implication of combining the above three properties is the following. At any point in time Condition C2 implies $\gamma(\omega,t,T)$ is continuous in T and Condition C3 states $\gamma(\omega,t,T) \neq 0$ for all $T \in [t,\infty)$. Consequently, at any point in time $\gamma(\omega,t,T)$ will have the same sign for all $T \in [t,\infty)$. Placing this restriction on equation (3) demonstrates that the infinite maturity forward rate never decreases over time which is consistent with a similar result obtained in Dybvig-Ingersoll-Ross (1996). The result of a non-decreasing evolution over time for the infinite maturity forward rate

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13 This obtains directly from equation (2) where $\tau = T - s$. Expressing equation (3) in terms of time to maturity τ ensures that Condition C6 (considered shortly) is well defined.

14 This result follows directly from equation (3) and the result that $\lim_{T \to \infty} \gamma(\omega,t,T) = 0$ for all $t \in [0,\infty)$.

15 This becomes clear when applying Itô's lemma to the forward rate curve which indicates that the forward rate volatility structure is related to the forward rate curve via $\gamma(\omega,t,T) = \left[\partial f(r(t),t,T) \,/\, \partial r(t)\right] \sigma(\omega,t)$, where $\sigma(\omega,t)$ is the volatility of $r(t)$. Condition C3 implies $\sigma(\omega,t) \neq 0$ but condition C5 implies $\lim_{T \to \infty} \gamma(\omega,t,T) = 0$. Hence $\partial f(r(t),t,\infty) \,/\, \partial r(t) = 0$ which is the required result.

implies $\lim_{t\to\infty}\lim_{\tau\to\infty} f(t,t+\tau)$ may become infinite, however, if it is desirable that $\lim_{t\to\infty}\lim_{\tau\to\infty} f(t,t+\tau)$ remains finite then the following condition must hold in addition to conditions C1 to C5[16],

Condition C6: $$\lim_{t\to\infty}\ \lim_{\tau\to\infty}\int_0^t\left[\gamma(\omega,s,s+\tau)\int_0^\tau\gamma(\omega,s,s+v)\,dv\right]ds \;=\; C(\omega)$$

where C(ω) is finite and may depend on the term structure's entire realization.

Condition C6 ensures that the infinite maturity forward rate does not grow without bound and hence some control as to the level of $f(t,\infty)$ can be obtained by ensuring that this bound is not "too large". For example, the volatility structure $\gamma(\omega,t,T) = \sigma\big/\sqrt{1+\kappa\,(T-t)}$ satisfies, where appropriate, conditions C1 to C5. However such a specification violates Condition C6, indicating that the infinite maturity forward rate does grow without bound over time. It must be noted that the forward rate volatility structure may depend on the realization of the term structure's evolution and hence Condition C6 cannot always be verified. However, Condition C6 can be verified when the volatility structure is deterministic, or if the volatility structure has the property $\left|\int_0^\infty\gamma(\omega,s,s+v)\,dv\right| < \infty$ for all $s \in [0,\infty]$ in which case $\lim_{t\to\infty}\lim_{\tau\to\infty} f(t,t+\tau) = f(0,\infty)$.

From the term structure properties considered, Property 3 intuitively seems to be the most restrictive for the following two reasons. First, the most that can be said about the asymptotic behavior of the term structure is that one dollar to be received an infinite time in the future must be worth nothing today. The only restriction this implies for any forward rate curve $f(t,T)$ is $\lim_{T\to\infty}\int_t^T f(t,v)\,dv = +\infty$. This does not imply that $\lim_{T\to\infty} f(t,T)$ is necessarily a constant at time t, in fact $f(t,T)$ may continually oscillate or approach infinity as maturity increases. Second, Property

[16] This condition follows directly from equation (3).

3 implies the non-intuitive result that the infinite maturity forward rate evolves in a locally deterministic fashion. In contrast, properties 1 and 2 do not restrict the term structure's behavior well beyond that which is plausible. Property 1 implies that forward rates which are "close" in maturity are of similar value[17] and Property 2 implies that forward rates evolve in an uncertain manner. Consequently the remainder of this section discusses violations of conditions C4 and C5 (the necessary conditions for Property 3) while maintaining conditions C1, C2 and C3 (the necessary conditions for properties 1 and 2).

Before proceeding with the discussion, it will be helpful to consider the relationship between the initial term structure, expected spot interest rates and risk preferences[18];

$$f(0,T) = E\left[r(T) - \int_0^T \gamma(\omega,s,T)\, \lambda(\omega,s)\, ds - \int_0^T \gamma(\omega,s,T) \int_s^T \gamma(\omega,s,v)\, dv\, ds \,\middle|\, \mathscr{F}_0 \right] \qquad \textbf{(4)}$$

where $\mathscr{F}_0$ represents the information available at time 0.

Equation (4) demonstrates that the initial forward rate with maturity T can be considered as a composition of the following three components. First, $E\left[r(T) \mid \mathscr{F}_0\right]$ which is the time zero expected spot interest rate for time T. Second, $E\left[\int_0^T \gamma(\omega,s,T)\, \lambda(\omega,s)\, ds \,\middle|\, \mathscr{F}_0\right]$ which is the risk premium required for fixing at time zero the instantaneous spot interest rate for time T. Third, $E\left[\int_0^T \gamma(\omega,s,T) \int_s^T \gamma(\omega,s,v)\, ds \,\middle|\, \mathscr{F}_0\right]$ which is an adjustment term resulting from expressing the term structure via forward rates instead of bond prices. Since our attention is restricted to volatility structures which satisfy conditions C2 and C3, this adjustment term is always positive.

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[17] Note that this only reflects continuity of the forward rate curve however differentiability of the forward rate curve is appealing for analytic tractability. Note, however, that many of the results discussed in this section have only relied upon continuity arguments.

[18] This results from equation (2) evaluated at $t = T$ and then taking expectations.

Some "plausible" economic behavior relating to the components of equation (4) are the following. First, an initial term structure must possess the property $\int_0^\infty f(0,v)\,dv = +\infty$ to ensure that an investor perceives one dollar to be received an infinite time in the future to be worth nothing today. Second, expected spot interest rates must be finite, otherwise it is expected that anyone with finite wealth cannot borrow over a fixed time interval. It is natural to extend this behavior to include the limit as time goes to infinity to ensure that our economy does not approach the situation where individuals expect not to be able to borrow. Third, it is usual to assume that individuals are risk averse but not infinitely so; this constrains the market price of risk to be negative and finite[19]. Again it is natural to extend this behavior to ensure that individuals remain risk averse asymptotically in time. Finally, Condition C3 implies that the spot interest rate volatility at time t, $\gamma(\omega,t,t)$, is non-zero for all $t \in [0,\infty)$. This should be extended to include the limit as time goes to infinity to prevent a term structure model implying that spot interest rate uncertainty is resolved asymptotically in time. The discussion regarding violations of conditions C4 and C5 is now continued.

Violations of Condition C4 can be partitioned into the following three categories: i) $\lim_{T\to\infty} f(0,T) = -\infty$; ii) $\lim_{T\to\infty} f(0,T) = +\infty$; or iii) $f(0,T)$ exhibits continued cyclic behavior as maturity T increases. The first category is clearly not plausible as such an initial term structure violates $\int_0^\infty f(0,v)\,dv = +\infty$. To obtain some intuition regarding the plausibility of the second category consider equation (4) when $\lim_{T\to\infty} f(0,T) = +\infty$, that is

$$\lim_{T\to\infty} E\left[\, r(T) - \int_0^T \gamma(\omega,s,T)\,\lambda(\omega,s)\,ds - \int_0^T \gamma(\omega,s,T) \int_s^T \gamma(\omega,s,v)\,dv\,ds \,\middle|\, \mathscr{F}_0 \right] = +\infty .$$

Given that individuals are risk averse but not infinitely so, let $-\infty < \underline{\lambda} \leq \lambda(\omega,t) < 0$ for all $t \in [0,\infty]$ and all possible realizations ω. Essentially $\underline{\lambda}$ represents the largest risk premium per unit of risk

[19] See Jeffrey (1995) Section II for an intuitive explanation of why the market price of risk, $\lambda(\omega,t)$, is negative for risk averse individuals. Furthermore, $|\lambda(\omega,t)| < \infty$ indicates that individuals only require a finite amount of compensation per unit of risk.

that individuals will ever demand in a given model. Further given the expected asymptotic spot interest rate remains finite, that is $| \lim_{t \to \infty} E\left[r(t) \mid \mathscr{F}_0 \right] | < +\infty$, the above equation indicates that the only means by which $\lim_{T \to \infty} f(0,T) = +\infty$ can occur is when the following equation is satisfied,

$$\lim_{T \to \infty} E\left[-\underline{\lambda} \int_0^T \gamma(\omega,s,T)\, ds \; - \int_0^T \gamma(\omega,s,T) \int_s^T \gamma(\omega,s,v)\, dv\, ds \,\middle|\, \mathscr{F}_0 \right] = +\infty \qquad \textbf{(5)}$$

Equation (5) and the restriction that the spot interest rate volatility should not approach zero as time approaches infinity, that is $| \lim_{t \to \infty} \gamma(\omega,t,t) | > 0$, impose severe necessary restrictions for a volatility structure to be consistent with both plausible economic behavior and $\lim_{T \to \infty} f(0,T) = +\infty$. For example, volatility structures which do not satisfy equation (5) include the HJM constant and exponential decay volatility structure models, Vasicek's (1977) model, and $\gamma(\omega,t,T) = \sigma / \sqrt{1 + \kappa\,(T-t)}$ where $\kappa > 0$. Given the severity of these restrictive necessary conditions I conjecture that an initial forward rate curve with the property $\lim_{T \to \infty} f(0,T) = +\infty$ is generally not plausible. Finally, in relation to the possibility that the initial forward rate curve may exhibit continued cyclic behavior as maturity increases, no arguments can be presented to suggest such an exclusion from plausible term structure models. Hence it is suggested that the only plausible behavior for the initial term structure is to either conform with Property 3, that is $\lim_{T \to \infty} f(0,T) = f(0,\infty)$ where $f(0,\infty)$ is a non-negative constant[20], or exhibit continued cyclic behavior as maturity T increases.

Now consider violations of Condition C5 while maintaining volatility structures satisfying conditions C2 and C3. Since these conditions imply that a volatility structure at any fixed point in time is continuous and non-zero, it is clear that $\gamma(\omega,t,T) \int_t^T \gamma(\omega,t,v)\, dv$ will always be positive. Further, if at time t the volatility structure does not have the property $\lim_{T \to \infty} \gamma(\omega,t,T) = 0$ then $\lim_{T \to \infty} \gamma(\omega,t,T) \int_t^T \gamma(\omega,t,v)\, dv = +\infty$ which implies that the drift of long term forward rates increases

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20 The non-negativity of $f(0,\infty)$ is necessary for $\lim_{T \to \infty} \int_0^T f(0,T) = +\infty$.

to infinity as maturity increases. Consequently, as long as the infinite maturity forward rate at time t is not infinitely negative and as long as the volatility structure does not have the property $\lim_{T\to\infty}\gamma(\omega,t,T) = 0$, then the infinite maturity forward rate at time $t+dt$ will be infinite. However, previously it has been argued that choosing an initial term structures with the property $\lim_{T\to\infty}|f(0,T)| = +\infty$ is not generally desirable. Redefining the forward rate curve at time $t+dt$ to be the initial term structure indicates that any model which implies $\lim_{T\to\infty} f(t+dt,T) = +\infty$ is not desirable. Consequently, Condition C5 should not generally be weakened.

IV. SUMMARY AND CONCLUSIONS

In this paper three properties exhibited by several existing term structure models are considered. Necessary conditions that result from these properties include i) differentiability of the initial forward rate curve and the forward rate volatility structure with respect to the maturity parameter, ii) the forward rate volatility structure is non-zero everywhere except for the limiting case as maturity goes to infinity where it is zero, and iii) the infinite maturity forward rate on the initial term structure exists and is a finite constant. A discussion regarding plausible economic behavior and the asymptotic behavior of the term structure suggests that these conditions should only be weakened to the extent of allowing the initial term structure to exhibit continued cyclic behavior as maturity increases. An analysis of the asymptotic behavior of the term structure's evolution, assuming the above necessary conditions hold, reveals the following. First, the dynamics of the infinite maturity forward rate are locally deterministic but fully deterministic if i) the forward rate volatility structure is deterministic, or ii) the forward rate volatility structure possesses the property $|\int_0^\infty \gamma(\omega,t,t+v)\,dv| < \infty$ for all $t \in [0,\infty)$, or iii) the term structure at time t is a deterministic function of time t, maturity T, and the spot interest rate at time t. Second, the infinite maturity forward rate cannot fall over time. This is consistent with a result derived by Dybvig-Ingersoll-Ross (1996).

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